MATH 4030 Differential Geometry
Problem Set 2
due 11/10/2017 (Wed) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use I, J to denote connected open intervals in $\mathbb{R}$.

1. Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve described in polar coordinates by $r = r(\theta)$, i.e.

$$\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta), \quad \theta \in I$$

   (a) Show that for any $[a, b] \subset I$, $L_a^b(\alpha) = \int_a^b \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta$.

   (b) Show that the curvature at $\theta \in I$ is given by

$$k(\theta) = \frac{2r'(\theta)^2 - r(\theta)r''(\theta) + r(\theta)^2}{[r'(\theta)^2 + r(\theta)^2]^{3/2}}.$$

2. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. with $k(s) > 0$ for all $s \in I$. Show that

   (a) α lies on a plane if and only if $\tau(s) \equiv 0$ for all $s \in I$.

   (b) α is an arc of a circular helix or circle if and only if both k and τ are constant.

3. Let $\alpha : I \to \mathbb{R}^3$ be a regular curve (not necessarily p.b.a.l.), show that the curvature and torsion of α at $t \in I$ is given respectively by

$$k(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} \quad \text{and} \quad \tau(t) = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{|\alpha'(t) \times \alpha''(t)|^2}.$$

4. (*Lancret's theorem*) A space curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l. with $k(s) > 0$ for all $s \in I$ is said to be a *helix* if all of its tangent lines make a constant angle with a given direction, i.e. there exists some unit vector $v \in \mathbb{R}^3$ such that $\alpha'(s) \cdot v \equiv$ constant. Prove that α is a helix if and only if there exists a constant $c \in \mathbb{R}$ such that $\tau(s) = ck(s)$ for all $s \in I$.

5. Show that the two-sheeted cone $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$ is not a surface.

6. If all normal lines to a connected surface S passes through a fixed point $p_0 \in \mathbb{R}^3$, show that S is contained in a sphere.

7. Let S be a compact surface, show that there exists a straight line ℓ cutting S orthogonally at at least two points. *(Hint: Consider two points on S which are of maximal distance apart.)*

8. Show that each of the subset $(a, b, c, \neq 0)$ below are surfaces

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = ax\},$$

$$S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = by\},$$

$$S_3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = cz\}.$$

Prove that they all intersect orthogonally.

Suggested Exercises

(no need to hand in)

1. Let $\alpha : I \to \mathbb{R}^2$ be a plane curve p.b.a.l.. Suppose that

$$|\alpha(s_0)| = \max_{s \in I} |\alpha(s)|$$

for some $s_0 \in I$. Prove that $|k(s_0)| \geq 1/|\alpha(s_0)|$. Can one say anything if "maximum" is replaced by "minimum"?

2. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. and $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ be a rigid motion of $\mathbb{R}^3$. If we let $\beta = \varphi \circ \alpha : I \to \mathbb{R}^3$, which is also a space curve p.b.a.l., show that $k_\beta(s) = k_\alpha(s)$ and

$$\tau_\beta(s) = \begin{cases} \tau_\alpha(s) & \text{if } \varphi \text{ is orientation-preserving,} \\ -\tau_\alpha(s) & \text{if } \varphi \text{ is orientation-reversing.} \end{cases}$$

3. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. with $k(s) > 0$, $k'(s) \neq 0$ and $\tau(s) \neq 0$ for all $s \in I$. Prove that the trace of α is contained in a sphere of radius $r > 0$ if and only if

$$\frac{1}{k(s)^2} + \frac{k'(s)^2}{k(s)^4 \tau(s)^2} \equiv r^2.$$

4. Let $\alpha : I \to \mathbb{R}^3$ be a space curve p.b.a.l. with $k(s) > 0$ and $\tau(s) \neq 0$ for all $s \in I$. We call α a *Bertrand curve* if there exists another space curve $\beta : I \to \mathbb{R}^3$ p.b.a.l. such that the normal lines of α and β coincide for each $s \in I$. In this case, we can write

$$\beta(s) = \alpha(s) + r(s) N_\alpha(s).$$

Show that $r(s) \equiv$ constant. Moreover, prove that α is a *Bertrand curve* if and only if

$$A k_\alpha(s) + B \tau_\alpha(s) \equiv 1$$

for some nonzero constants $A, B \in \mathbb{R}$.

5. Let $F(x, y, z) = z^2$. Prove that 0 is not a regular value of F but $F^{-1}(0)$ is a surface.

6. Show that $S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 - y^2\}$ is a surface and that the following are parametrizations for S:

$$X_1(u, v) = (u + v, u - v, 4uv), \quad (u, v) \in \mathbb{R}^2$$
$$X_2(u, v) = (u \cosh v, u \sinh v, u^2), \quad (u, v) \in \mathbb{R}^2, u \neq 0.$$

7. Determine the tangent planes of the surface $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ at the points $(x, y, 0) \in S$ and show that they are all parallel to the z-axis.

8. Let $S = \{p \in \mathbb{R}^3 : |p|^2 - \langle p, a\rangle^2 = r^2\}$ with $|a| = 1$ and $r > 0$ be a right cylinder of radius r whose axis is the line passing through the origin with direction a. Prove that

$$T_p S = \{v \in \mathbb{R}^3 : \langle p, v\rangle - \langle p, a\rangle\langle a, v\rangle = 0\}.$$

Conclude that all the normal lines of S cut the axis orthogonally. Prove the converse as well: i.e. if S is a connected surface whose normal lines all intersect a fixed straight line $\ell \subset \mathbb{R}^3$ orthogonally, then S is a subset of a right cylinder with axis ℓ.

9. Construct an explicit diffeomorphism between the ellipsoid $S_1 = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ and the sphere $S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.

10. Let $S \subset \mathbb{R}^3$ be a surface. Suppose $P \subset \mathbb{R}^3$ is a plane such that S lies on one side of P, show that $T_q P = T_q S$ at all $q \in P \cap S$.